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LINEAR ALGEBRA AND ITS APPLICATIONS

A bound for the condition of a hyperbolic eigenvector matrix

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Abstract

The hyperbolic eigenvector matrix is a matrix X which simultaneously diagonalizes the pair (H,J), where H is Hermitian positive definite and $J = \mathrm{diag}(\pm 1)$ such that $X^*HX = \Delta$ and $X^*JX = J$. We prove that the spectral condition of X, $\kappa(X)$, is bounded by $\kappa(X) \leqslant \sqrt{\min \kappa(D^*HD)}$, where the minimum is taken over all non-singular matrices D which commute with J. This bound is attainable and it can be simply computed. Similar results hold for other signature matrices J, like in the discretized Klein–Gordon equation. © 1999 Published by Elsevier Science Inc. All rights reserved.

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1. Introduction

We are considering the hyperbolic eigenvalue problem

$$Hx = \lambda Jx, \tag{1}$$

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where H is a $n \times n$ Hermitian positive definite matrix, and $J = \mathrm{diag}(\pm 1)$. There always exists a matrix X such that

$$X^*HX = \Delta, \qquad X^*JX = J, \tag{2}$$

where Δ is diagonal positive definite matrix. Since H is positive definite, the pair (H, J) is regular by definition from Ref. [9, Definition VI.1.2], so the existence of X follows from Ref. [9, Theorem VI.1.15, Corollary VI.1.19]. The matrix X is also called J-unitary. Obviously, the ith eigenvalue of the problem (1) is given by

$$\lambda_i = \Delta_{ii}J_{ii},$$

and the ith column of X is the corresponding eigenvector. We call such eigenvectors hyperbolic, or J-unitary, contrary to the standard unitary eigenvectors of the problem $H\mathbf{x} = \lambda\mathbf{x}$. The matrix X is also called a hyper-exchange matrix with respect to the signature matrix J [5].

The matrix X also appears in other linear algebra problems. For example, X is the eigenvector matrix of the matrix JH:

$$X^{-1}(JH)X = JX^*J(JH)X = J\Delta.$$

Also, X is right singular vector matrix of the *hyperbolic singular value decomposition* (HSVD) of the pair (G, J). The HSVD for the full column-rank G is defined as

$$G = U\Sigma X^*,$$

where

$$U^*U = I, \quad X^*JX = J, \quad \Sigma = \mathrm{diag}(\sigma_i), \quad \sigma_i > 0.$$

Such HSVD is used in the highly accurate algorithm for the eigenvalue decomposition of a possibly indefinite symmetric (Hermitian) matrix A [11,7]: the idea is to factorize A as $A = GJG^*$ [8] and then compute the HSVD of the pair (G, J). Further, HSVD and its variant for the full row-rank G is a suitable way to compute the eigenvalue decomposition of the difference of two outer products [15,5], and the condition of X appears in the perturbation bounds for the eigenvalues of the non-singular matrix GJG^* [13]. Also, note that hyperbolic eigenvalue problems with other signature matrices (cf. Section 3) arise within some Lanczos-type algorithms for non-symmetric matrices [4].

In this paper $\|\cdot\|$ denotes the spectral matrix norm, and $\kappa(A)$ denotes the condition of a non-singular matrix A,

$$\kappa(A) = \|A\|\|A^{-1}\|.$$

The hyperbolic eigenvector matrix has two important properties.

1. All matrices which perform the simultaneous diagonalization (2) have the same condition [11].

2. $\kappa(X) = \|X\|^2$. Moreover, the singular values of X come in pairs of reciprocals, $\{\sigma, 1/\sigma\}$.

The condition $\kappa(X)$ can be expressed in terms of a Hermitian matrix which is associated to the problem (1). Let us define the spectral absolute value $|A|_S$ of the Hermitian matrix A as its positive definite polar factor. That is, if $A = Q\Lambda Q^*$ is the eigenvalue decomposition of A, then

$$|A|_S = Q|\Lambda|Q^* = \sqrt{A^2}.$$

Theorem 1. *Let $H = Z^*Z$ be some factorization of H. Then*

$$\kappa(X) = \max_{x\neq 0} \frac{\mathbf{x}ZZ^*\mathbf{x}}{\mathbf{x}^*|ZJZ^*|_S\mathbf{x}} = \max_{x\neq 0} \frac{\mathbf{x}^*|ZJZ^*|_S\mathbf{x}}{\mathbf{x}ZZ^*\mathbf{x}}.$$

Proof. The first equality was proved in Ref. [13], and the second equality follows because the eigenvalues of XX^* come in the pairs of reciprocals. □

Note that the spectral absolute value appears naturally in the relative perturbation bounds for Hermitian and normal matrices [14,1].

Since the maxima in Theorem 1 are not easy to compute, it is of interest to obtain a simpler bound for $\kappa(X)$. Veselić [12] recently proved that

$$\kappa(X) \leqslant \min_{D\in\mathscr{D}} \kappa(D^*HD),$$

where $\mathscr{D}$ is the set of all non-singular matrices which commute with J. In this paper we shall prove a better bound, namely

$$\kappa(X) \leqslant \sqrt{\min_{D\in\mathscr{D}} \kappa(D^*HD)}. \tag{3}$$

We shall also show for which matrices D the minimum is attained, and for which matrices H the bound itself is attained.

The rest of the paper is organized as follows: in Section 2 we prove the above results, and in Section 3 we apply our results to eigenvalue problems with other signature matrices, and in particular to the discretized Klein–Gordon equation and some Hamiltonian systems.

2. Bound for $\kappa(X)$

We shall prove the bound (3) for $\kappa(X)$ in two stages: we shall first analyze the case when the bound is an equality, and then prove the bound itself. From now on we assume without loss of generality that J has the form

$$J = \begin{bmatrix} I_m & 0 \\ 0 & -I_{n-m} \end{bmatrix} \tag{4}$$

which is easily achieved by permutation. Since all results of this section are trivial if in $m = 0$ or $m = n$, we assume that $0 < m < n$. Also,

$$\mathscr{D} = \{D = D_1 \oplus D_2 : D_1 \in C^{m,m}, D_2 \in C^{n-m,n-m}, \text{ non-singular}\}.$$

will denote the set of all non-singular matrices which commute with J from Ref. [4]. To prove our results we need the following theorem which appeared in Ref. [3] (see also Ref. [2]).

Theorem 2 [3, Theorem 2]. *Let*

$$H = \begin{bmatrix} I_m & \Psi \\ \Psi^* & I_{n-m} \end{bmatrix} \tag{5}$$

be positive definite. Then

$$\kappa(H) = \min_{D \in \mathscr{D}} \kappa(D^* H D).$$

Theorem 3 shows that the bound (3) becomes an equality for matrices of the form (5).

Theorem 3. *Let J and H be given by* (4) *and* (5), *respectively. Let X be some matrix which diagonalizes the pair (H, J) according to* (2). *Then*

$$\kappa(X) = \sqrt{\kappa(H)} = \sqrt{\min_{D \in \mathscr{D}} \kappa(D^* H D)}.$$

Proof. The second equality follows from Theorem 2. Let us construct one particular X. Let $U^* \Psi V = \Sigma = \operatorname{diag}(\sigma_i)$ be the singular value decomposition of Ψ. Set

$$W = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix},$$

and $H_1 = W^* H W$. Then $W^* J W = J$ and

$$H_1 = \begin{bmatrix} I_m & \Sigma \\ \Sigma^T & I_{n-m} \end{bmatrix}.$$

Since H is positive definite we have $\sigma_i \leqslant \|\Psi\| < 1$, and

$$\kappa(H_1) = \kappa(H) = \frac{1 + \sigma_{\max}}{1 - \sigma_{\max}}.$$

Let R be the matrix which diagonalizes the pair (H_1, J) according to Eq. (2), and let $k = \min\{m, n-m\}$. Then

$$R = \begin{bmatrix}
c_1 & & & & & s_1 & & & & \\
& c_2 & & & & & s_2 & & & \\
& & \ddots & & & & & \ddots & & \\
& & & c_k & & & & & s_k & \\
& & & & I_{m-k} & & & & & 0 \\
s_1 & & & & & c_1 & & & & \\
& s_2 & & & & & c_2 & & & \\
& & \ddots & & & & & \ddots & & \\
& & & s_k & & & & & c_k & \\
& & & & 0 & & & & & I_{n-m-k}
\end{bmatrix}. \tag{6}$$

Here, c_i and s_i are hyperbolic sines and cosines computed as follows: if $\sigma_i = 0$, then $c_i = 1$, $s_i = 0$; otherwise

$$t_i = -\frac{\sigma_i}{1 + \sqrt{1 - \sigma_i^2}},$$

$$c_i = \frac{1}{\sqrt{1 - t_i^2}},$$

$$s_i = c_i \cdot t_i.$$

Now we have $X = WR$, and, since W is unitary $\kappa(X) = \kappa(R)$. A striaghtforward computation shows that

$$\kappa(R) = \frac{1 + \max\ |t_i|}{1 - \min\ |t_i|} = \sqrt{\frac{1 + \sigma_{\max}}{1 - \sigma_{\max}}} = \sqrt{\kappa(H)}.$$

The theorem now follows from the fact that all X which perform the required diagonalization have the same condition. □

The aim of our main theorem is twofold: to prove the bound (3) and to define the matrix D for which the minimum is attained.

Theorem 4. *Let J be given by Eq.* (4) *and let*

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^* & H_{22} \end{bmatrix} \tag{7}$$

be partitioned accordingly. Let

$$D = \begin{bmatrix} G_1^{-1} & 0 \\ 0 & G_2^{-1} \end{bmatrix},$$

where $H_{ii} = G_i^* G_i$ is some factorization of H_{ii} for $i \in \{1, 2\}$, respectively. Let $\hat{H} = D^* H D$ and let X be the matrix which diagonalizes the pair (H, J) according to Eq. (2). Then

$$\kappa(X) \leqslant \sqrt{\kappa(\hat{H})} = \sqrt{\min_{D \in \mathscr{D}} \kappa(D^* H D)}. \tag{8}$$

Proof. The equality in Eq. (8) follows from Theorem 3 since $\hat{H}$ has the form (5) with $\Psi = G_1^{-*} H_{12} G_2^{-1}$. Since all X which perform the diagonalization (2) have the same condition, it remains to prove the inequality in Eq. (8) for one particular X which we shall now construct. Let $\hat{X} = WR$ be the matrix which diagonalizes the pair $(\hat{H}, J)$ as in the proof of Theorem 3, and let $k = \min\{m, n, -m\}$. Then

$$\hat{X}^* \hat{H} \hat{X} = \text{ diag } = S^2 = T^2 \oplus I_{m-k} \ominus T^2 \oplus I_{n-m-k},$$

where $T^2 = \operatorname{diag}(1 + \sigma_i \cdot t_i)$. Special forms of R from Eq. (6) and S imply that they commute. Set

$$Z = DWRS^{-1}. \tag{9}$$

Since $Z^* H Z = I$ we conclude that the eigenvalue decomposition of the matrix $Z^* J Z$ is given by

$$Z^* J Z = Q J \Delta^{-1} Q^*, \tag{10}$$

where Q is unitary, and Δ is given by Eq. (2). Therefore, the matrix

$$X = Z Q \Delta^{1/2}$$

performs the required diagonalization of the pair (H, J). Since $\kappa(X) = \|X\|^2$ we have

$$\kappa(X) = \lambda_{\max}(Z Q \Delta Q^* Z^*). \tag{11}$$

Inverting Eq. (10) gives

$$Z^{-1} J Z^{-*} = Q \Delta J Q^* = Q \Delta Q^* Q J Q^*,$$

and inserting this expression for $Q \Delta Q^*$ into Eq. (11) gives

$$\kappa(X) = \lambda_{\max}(Z Z^{-1} J Z^{-*} Q J Q^* Z^*) = \lambda_{\max}(Q J Q^* Z^* J Z^{-*}). \tag{12}$$

In the last equality we have used the fact that for any square matrices A and B, the matrices AB and BA have the same eigenvalues. From Eq. (9), by using J-orthogonality of R, the fact that R and S commute, and the fact that D, W and S commute with J, we have

$$Z^*JZ^{-*} = S^{-1}R^*W^*D^*JD^{-*}W^{-*}R^{-*}S = R^*JR^{-*} = R^2J.$$

By inserting this into Eq. (12), by using the fact that $\lambda_i(A) \leqslant \|A\|$ for any matrix A, and by using unitarity of Q and J, Theorem 3 finally gives

$$\kappa(X) = \lambda_{\max}(QJQ^*R^2J) \leqslant \|QJQ^*R^2J\| = \|R^2\| = \kappa(R) = \kappa(\hat{X}) = \sqrt{\kappa(\hat{X})}$$

as desired. □

3. Other signature matrices

Theorem 4 yields as a corollary similar results for the eigenvalue problems with other Hermitian and skew-Hermitian signature matrices. Let us consider the simultaneous diagonalization of the pair (H, J_S), where H is positive definite matrix and the signature matrix is given by

$$J_S = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \quad \text{or} \quad J_S = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

In both the cases we seek X such that $X^*J_SX = J_S$ and

$$X^*HX = \begin{bmatrix} \Delta & \Delta_1 \\ \Delta_1 & \Delta \end{bmatrix} \quad \text{or} \quad X^*HX = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}, \tag{13}$$

respectively, where Δ and Δ_1 are diagonal matrices. These forms readily contain the eigenvalues of the problem

$$H\mathbf{x} = \lambda J_S \mathbf{x} \tag{14}$$

and keep real arithmetic whenever possible. Let us set

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I \end{bmatrix} \quad \text{or} \quad U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & iI \\ iI & I \end{bmatrix},$$

respectively, where i is the imaginary unit. Then we have

$$U^*J_SU = J \quad \text{or} \quad U^*J_SU = iJ,$$

respectively, where $J = I \oplus (-I)$. Now set $\tilde{H} = U^*HU$, and let $\tilde{X}$ be the matrix which diagonalizes the pair $(\tilde{H}, J)$ as in Eq. (2) such that $\tilde{X}^*J\tilde{X} = J$ and

$$\tilde{X}^*\tilde{H}\tilde{X} = \begin{bmatrix} \Delta + \Delta_1 & 0 \\ 0 & \Delta - \Delta_1 \end{bmatrix} \quad \text{or} \quad \tilde{X}^*\tilde{H}\tilde{X} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix},$$

respectively. Then $X = U\tilde{X}U^*$ performs the diagonalization (13). and, by applying Theorem 4 to the above reduction, we have

$$\kappa(X) = \kappa(\tilde{X}) \leqslant \sqrt{\min_{D \in \mathcal{D}} \kappa(D^*HD)}, \tag{15}$$

where $\tilde{\mathscr{D}}$ is the set of all non-singular matrices which commute with the respective J_S.

The reduction Eq. (13) for the first choice of J_S appears in the case of the discretized Klein–Gordon equation (cf. [10]) which consists of the quadratic eigenvalue problem

$$(\lambda^2 - 2\lambda V + V^2 - Z^2)\psi = 0,$$

where Z and V are real symmetric matrices, Z is positive definite, and $\|VZ^{-1}\| < 1$. This eigenvalue problem is equivalent to the problem (14) with

$$H = \begin{bmatrix} L^*L & L^{-1}VL \\ L^*VL^{-*} & L^*L \end{bmatrix}, \qquad J_S = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

where $Z = LL^*$ is some factorization of Z. By taking

$$D_0 = \begin{bmatrix} L^{-1} & 0 \\ 0 & L^{-1} \end{bmatrix}$$

which commutes with J_S, the relation (15) implies that the condition of the matrix X which performs the required diagonalization of the pair (H, J_S) is bounded by

$$\kappa(X) \leqslant \sqrt{\kappa(D_0^* H D_0)} = \sqrt{\frac{1 + \|VZ^{-1}\|}{1 - \|VZ^{-1}\|}}.$$

The reduction Eq. (13) for the second choice of J_S comes in solution of certain Hamiltonian systems. It is also a part of the highly accurate eigenvalue decomposition algorithm for skew-symmetric matrices [6]. If H is partitioned according to this J_S as in Eq. (7), then the minimum in Eq. (15) is attained for

$$D = \begin{bmatrix} G_1 + G_2 & -i(G_1 - G_2) \\ i(G_1 - G_2) & G_1 + G_2 \end{bmatrix},$$

where

$$G_1^* G_1 = \tfrac{1}{2} \times [H_{11} + H_{22} + i(H_{12} - H_{12}^*)],$$

$$G_2^* G_2 = \tfrac{1}{2} \times [H_{11} + H_{22} + i(H_{12}^* - H_{12})],$$

and Eq. (15) is an equality if $H_{11} + H_{22} = 2I$ and $H_{12} = H_{12}^*$.

Similar bounds can be easily derived for a matrix X which diagonalizes any pair (H, J), where H is positive definite and J satisfies merely the condition

$$J = J^* = J^{-1} \quad \text{or} \quad J = -J^* = -J^{-1}.$$

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